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J. Phys. A: Math. Gen. 26 (1993) 4723–4736. Printed in the UK

Solutions of Yang–Baxter equations with colours: standard and non-standard

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Received 23 November 1992

Abstract. Colour-extended solutions of YBE associated with $G=B_n$, C_n and D_n are derived and the coloured Birman–Wenzl algebra (CB-WA) is established. We point out that the CB-WA still satisfies the redundance conditions of Murakami [1]. For non-standard solution, we take the non-trivial colour-solution of D_2 as an example. We find that the colour-constraints induce a new solution with $q^4=1$, the implication of this is not clear to us at the present.

1. Introduction

Recently, solutions of the coloured Yang–Baxter equations (YBE) and the corresponding link polynomials have attracted much attention [1–5]. As an extension, a coloured braid group representation (CBGR) satisfies the equations

$$\check{R}_{12}(\lambda,\mu)\check{R}_{23}(\lambda,\nu)\check{R}_{12}(\mu,\nu)=\check{R}_{23}(\mu,\nu)\check{R}_{12}(\lambda,\nu)\check{R}_{23}(\lambda,\mu) \tag{1.1}$$

where λ, μ and ν stand for colour-parameters. The simplest example is 4 by 4 solutions for which two types of solution have been obtained, namely a standard solution [6]

$$\check{R}_I(\lambda,\mu)=\begin{bmatrix} q & & & \\ & 0 & \eta X(\lambda) & \\ & \eta^{-1}X^{-1}(\mu) & (q-q^{-1})g(\lambda)g^{-1}(\mu) & \\ & & & qX(\lambda)X^{-1}(\mu)\end{bmatrix} \tag{1.2}$$

and non-standard solution

$$\check{R}_{II}(\lambda,\mu)=\begin{bmatrix} q & & & \\ & 0 & \eta X(\lambda) & \\ & \eta^{-1}Y(\mu) & W(\lambda,\mu) & \\ & & & -q^{-1}X(\lambda)Y(\mu)\end{bmatrix} \tag{1.3}$$

where $X(\lambda)$, $g(\lambda)$, $Y(\mu)$ are arbitrary colour-dependent parameters and $W(\lambda,\mu)$ satisfies the relation

$$W(\lambda,\mu)W(\mu,\nu)=\{q-q^{-1}X(\mu)Y(\mu)\}W(\lambda,\nu). \tag{1.4}$$

Note that (1.3) contains Marakami's solution [1, 2] as a special case.

Obviously, equation (1.2) is the colour-extension of the usual standard 4 by 4 solution associated with SU(2), whereas (1.3) is the colour-extension of the corresponding non-standard solution [7, 8]. However, it turns out that both of them satisfy the quantum double of Drinfeld [9] and η and $g(\lambda)$ are related to gauge transformations. From the point of view of representation of quantum algebra, (1.2) is related to the case where q is generic, whilst (1.3) is for q being a root of unity [3, 10]. However, for CBGR associated with the fundamental representations of $G=B_n$, D_n, C_n, the situation is not so clear because the explicit form of the quantum double for G is very difficult to write down. If we follow the strategy of [3, 10] to discuss the problem then the q-boson realization theory of quantum algebra and the explicit R-operator forms for G are complicated and unsatisfactory.

In this paper we construct explicit forms of the standard solutions of equation (1.1) for G and their coloured Birman–Wenzl algebraic (CB–WA) structure, the latter being the colour-extension of the usual Birman–Wenzl algebra (B–WA). The 'redundance' property of B–WA proved in [1, 19] will also be extended to the CB–WA, that is, the CB–WA still obeys the Murakami redundance conditions [1]. For the non-standard solution of (1.1) we shall give an example associated with non-trivial BGR of D_2, but it turns out to be a completely different picture from $SU_q(2)$.

2. Coloured standard BGR for G

Let G be the fundamental representations of B_n, C_n and D_n; the $\check{R}(\lambda, \mu)$ associated with G can be written in the form

$$(\check{R}(\lambda, \mu))^{ab}_{cd} = u_a(\lambda, \mu)\delta_{abcd} + p^{(a,b)}(\lambda, \mu)\delta_{ad}\delta_{bc}\big|_{a\neq b} + W^{(a,b)}(\lambda, \mu)\delta_{ac}\delta_{bd}\big|_{a<b} + q^{(a,c)}(\lambda, \mu)\delta_{a-b}\delta_{c-d}\big|_{\substack{a\neq \pm c \\ |a|<|c| \text{ if } c<0 \\ |c|<|a| \text{ if } a<0}} \tag{2.1}$$

where $\delta_{abcd}=1$ for $a=b=c=d$, 0 otherwise, and $\delta_{ab}=1$ for $a=b$, 0 otherwise. The labels a, b, c, d, above assume the following values

$$\left[-\left(\frac{N-1}{2}\right), -\left(\frac{N-1}{2}\right)+1, \ldots, \left(\frac{N-1}{2}\right)\right] \tag{2.2}$$

$$(N=2n+1 \text{ for } B_n,\ N=2n \text{ for } C_n \text{ and } D_n).$$

Obviously, the $\check{R}(\lambda, \mu)$ has the same matrix structure form as the $\check{R}$-matrix without colour for G[11]. The only difference is that all the four unknown parameters $u_a(\lambda, \mu)$, $p^{(a,b)}(\lambda, \mu)$, $q^{(a,c)}(\lambda, \mu)$ and $w^{(a,b)}(\lambda, \mu)$ are now colour-dependent, where λ and μ stand for colours.

On substituting (2.1) into (1.1), we can find relations satisfied by the unknown parameters of (2.1). The computation is elementary but tedious and is similar to those performed in [11]. Omitting the details, we give the final results. Generally, we have ($w=q-q^{-1}$)

$$\begin{aligned} u_a(\lambda, \mu) &= qX_a(\lambda)X_a^{-1}(\mu) && (a\neq 0) \\ p^{(a,b)}(\lambda, \mu) &= X_b(\lambda)X_a^{-1}(\mu) && (a+b\neq 0) \\ w^{(a,b)}(\lambda, \mu) &= wX_b(\lambda)X_a^{-1}(\mu) && (a+b\neq 0) \end{aligned} \tag{2.3}$$

and other relations are: for B_n

$$
\begin{aligned}
&u_0(\lambda,\mu)=X_0(\lambda)X_0^{-1}(\mu) \qquad p^{(a,-a)}(\lambda,\mu)=q^{-1}X_{-a}(\lambda)X_a^{-1}(\mu)\\
&w^{(-a,a)}(\lambda,\mu)=w(1-q^{-2a+1})X_a(\lambda)X_a^{-1}(\mu)\\
&q^{(0,-a)}(\lambda,\mu)=-wq^{-a+1/2}X_0(\lambda)X_a^{-1}(\mu) \qquad (a>0)\\
&q^{(-a,0)}(\lambda,\mu)=-wq^{-a+1/2}X_a(\lambda)X_0^{-1}(\mu) \qquad (a>0)\\
&q^{(a,-b)}(\lambda,\mu)=-wq^{a-b}X_{-b}(\lambda)X_a^{-1}(\mu) \qquad (0<a<b)\\
&q^{(-b,a)}(\lambda,\mu)=-wq^{a-b}X_b(\lambda)X_{-a}^{-1}(\mu) \qquad (0<a<b)\\
&q^{(-a,-b)}(\lambda,\mu)=-wq^{-a-b+1}X_a(\lambda)X_b^{-1}(\mu) \qquad (a,b<0)
\end{aligned}
\tag{2.4}
$$

for C_n and D_n

$$
\begin{aligned}
&p^{(a,-a)}(\lambda,\mu)=q^{-1}X_{-a}(\lambda)X_a^{-1}(\mu)\\
&q^{(a,-b)}(\lambda,\mu)=-wq^{a-b}X_{-a}(\lambda)X_b^{-1}(\lambda) \qquad (0<a<b)\\
&q^{(-b,a)}(\lambda,\mu)=-wq^{a-b}X_b(\lambda)X_{-a}^{-1}(\mu) \qquad (0<a<b)
\end{aligned}
\tag{2.5}
$$

$$
w^{(-a,a)}(\lambda,\mu)=\begin{cases} w(1+q^{-2a-1})X_a(\lambda)X_a^{-1}(\mu) & \text{for } C_n\\ w(1-q^{-2a+1})X_a(\lambda)X_a^{-1}(\mu) & \text{for } D_n\end{cases}
\tag{2.6}
$$

$$
q^{(-a,-b)}(\lambda,\mu)=\begin{cases} wq^{-a-b-1}X_a(\lambda)X_b^{-1}(\mu) & \text{for } C_n\\ -wq^{-a-b+1}X_a(\lambda)X_b^{-1}(\mu) & \text{for } D_n.\end{cases}
\tag{2.7}
$$

In the above, $X_a(\lambda)$s are arbitrary colour-dependent parameters and $X_a(\lambda)X_{-a}(\lambda)$ is independent of the index a.

By taking $\lambda=\mu$ the above solutions are reduced to the known BGRs given by Jimbo [12]. For the non-coloured case it is known that the BGR for G leads to B-WA [13, 14]. To construct the coloured B-WA, new matrices are introduced

$$
I(\lambda,\mu)=\sum_{a,b} X_a(\lambda)X_b^{-1}(\mu)e_{aa}\otimes e_{bb}
\tag{2.8}
$$

where $(e_{ab})_{cd}=\delta_{ac}\delta_{bd}$, and

$$
\begin{aligned}
&E(\lambda,\mu)=m^{-1}\{\check{R}(\lambda,\mu)+\check{R}^{-1}(\mu,\lambda)\}-I(\lambda,\mu)\\
&m=\mathrm{i}(q-q^{-1})=\mathrm{i}w.
\end{aligned}
\tag{2.9}
$$

Defining

$$
I_i(\lambda,\mu)=I\otimes\ldots\otimes I\otimes \underset{i,\,i+1}{I(\lambda,\mu)}\otimes I\otimes\ldots
\tag{2.10}
$$

$$
E_i(\lambda,\mu)=I\otimes\ldots\otimes I\otimes \underset{i,\,i+1}{E(\lambda,\mu)}\otimes I\otimes\ldots
\tag{2.11}
$$

$$
b_i(\lambda,\mu)=I\otimes\ldots\otimes I\otimes \underset{i,\,i+1}{\check{R}(\lambda,\mu)}\otimes I\otimes\ldots
\tag{2.12}
$$

we can now extend the usual B-WA to the coloured B-WA. It turns out that the basic relations are (no sum over the repeated indices)

$$
\begin{aligned}
&b_i(\lambda,\mu)b_{i+1}(\lambda,\nu)b_i(\mu,\nu)=b_{i+1}(\mu,\nu)b_i(\lambda,\nu)b_{i+1}(\lambda,\mu)\\
&b_i(\lambda,\mu)b_j(\nu,\rho)=b_j(\nu,\rho)b_i(\lambda,\mu) \qquad |i-j|\geqslant 2
\end{aligned}
\tag{2.13}
$$

$$I_i(\lambda, \mu)I_{i+1}(\lambda, \nu)I_i(\mu, \nu) = I_{i+1}(\mu, \nu)I_i(\lambda, \nu)I_{i+1}(\lambda, \mu)$$

$$I_i(\lambda, \mu)I_j(\nu, \rho) = I_j(\nu, \rho)I_i(\lambda, \mu) \qquad |i-j| \geqslant 2 \tag{2.14}$$

$$I_i(\lambda, \mu)I_i(\mu, \lambda) = I \qquad I_i(\lambda, \lambda) = I$$

$$E_i(\lambda, \mu) = m^{-1}\{b_i(\lambda, \mu) + b_i^{-1}(\mu, \lambda)\} - I_i(\lambda, \mu) \tag{2.15}$$

$$b_i(\lambda, \mu)I_i(\mu, \lambda) = I_i(\lambda, \mu)b_i(\mu, \lambda) \tag{2.16}$$

$$\begin{aligned} b_i(\lambda, \mu)I_{i+1}(\lambda, \nu)I_i(\mu, \nu) &= I_{i+1}(\mu, \nu)b_i(\lambda, \nu)I_{i+1}(\lambda, \mu) \\ &= I_i(\lambda, \mu)I_{i+1}(\lambda, \nu)b_i(\mu, \nu) \\ b_{i+1}(\mu, \nu)I_i(\lambda, \nu)I_{i+1}(\lambda, \mu) &= I_i(\lambda, \mu)b_{i+1}(\lambda, \nu)I_i(\mu, \nu) \\ &= I_{i+1}(\mu, \nu)I_i(\lambda, \nu)b_{i+1}(\lambda, \mu) \end{aligned} \tag{2.17}$$

$$\begin{aligned} E_i(\lambda, \mu)b_{i+1}(\lambda, \nu)E_i(\mu, \nu) &= lI_{i+1}(\mu, \nu)E_i(\lambda, \nu)I_{i+1}(\lambda, \mu) \\ E_{i+1}(\mu, \nu)b_i(\lambda, \nu)E_{i+1}(\lambda, \mu) &= lI_i(\lambda, \mu)E_{i+1}(\lambda, \nu)I_i(\mu, \nu) \end{aligned} \tag{2.18}$$

$$E_i(\lambda, \mu)b_i(\mu, \lambda) = l^{-1}E_i(\lambda, \mu)I_i(\mu, \lambda) \qquad l = -\mathrm{i}\lambda^{-1} \tag{2.19}$$

$$\lambda = \begin{cases} q^{-2n} & \text{for } B_n \\ -q^{-2n-1} & \text{for } C_n \\ q^{-2n+1} & \text{for } D_n \end{cases} \tag{2.20}$$

and from the above relations, we derive the following relations for CB-WA

$$\begin{aligned} E_i(\lambda, \mu)E_{i+1}(\lambda, \nu)E_i(\mu, \nu) &= I_{i+1}(\mu, \nu)E_i(\lambda, \nu)I_{i+1}(\lambda, \mu) \\ &= I_i(\lambda, \mu)I_{i+1}(\lambda, \nu)E_i(\mu, \nu) \\ E_{i+1}(\mu, \nu)E_i(\lambda, \nu)E_{i+1}(\lambda, \nu) &= I_i(\lambda, \mu)E_{i+1}(\lambda, \nu)I_i(\mu, \nu) \\ &= E_{i+1}(\mu, \nu)I_i(\lambda, \nu)I_{i+1}(\mu, \nu) \end{aligned} \tag{2.21}$$

$$\begin{aligned} b_i(\lambda, \mu)b_{i+1}(\lambda, \nu)E_i(\mu, \nu) &= E_{i+1}(\mu, \nu)b_i(\lambda, \nu)b_{i+1}(\lambda, \mu) \\ E_i(\lambda, \mu)b_{i+1}(\lambda, \nu)b_i(\mu, \nu) &= b_{i+1}(\mu, \nu)b_i(\lambda, \nu)E_{i+1}(\lambda, \mu) \end{aligned} \tag{2.22}$$

$$E_i(\lambda, \mu)I_i(\mu, \lambda) = I_i(\lambda, \mu)E_i(\mu, \lambda) \tag{2.23}$$

$$E_i(\lambda, \mu)E_i(\mu, \lambda) = \{m^{-1}(l + l^{-1}) - 1\}E_i(\lambda, \mu)I_i(\mu, \lambda) \tag{2.24}$$

$$b_i(\lambda, \mu)b_i(\mu, \lambda) = m\{b_i(\lambda, \mu) + l^{-1}E_i(\lambda, \mu)\}I_i(\mu, \lambda) + I \tag{2.25}$$

$$\begin{aligned} &b_i(\lambda, \mu)b_i(\mu, \lambda)b_i(\lambda, \mu) \\ &\quad = (m + l^{-1})b_i(\lambda, \mu)b_i(\mu, \lambda)I_i(\lambda, \mu) - (l^{-1}m + 1)b_i(\lambda, \mu) + l^{-1}I_i(\lambda, \mu) \end{aligned} \tag{2.26}$$

$$b_i(\lambda, \mu)E_{i+1}(\lambda, \nu)E_i(\mu, \nu) = b_{i+1}(\nu, \mu)E_i(\lambda, \nu)I_{i+\lambda}(\lambda, \mu) \tag{2.27}$$

$$\begin{aligned} b_i(\lambda, \mu)E_{i+1}(\lambda, \nu)b_i(\mu, \nu) &= b_{i+1}^{-1}(\nu, \mu)E_i(\lambda, \nu)b_{i+1}^{-1}(\mu, \lambda) \\ b_{i+1}(\mu, \nu)E_i(\lambda, \nu)b_{i+1}(\lambda, \mu) &= b_i^{-1}(\mu, \lambda)E_{i+1}(\lambda, \nu)b_i^{-1}(\nu, \mu) \end{aligned} \tag{2.28}$$

$$\begin{aligned} E_i(\lambda, \mu)I_{i+1}(\lambda, \nu)E_i(\mu, \nu) &= \{m^{-1}(l + l^{-1}) - 1\}E_i(\lambda, \mu)I_{i+1}(\lambda, \nu)I_i(\mu, \nu) \\ E_{i+1}(\lambda, \mu)I_i(\lambda, \mu)E_{i+1}(\lambda, \nu) &= \{m^{-1}(l + l^{-1}) - 1\}I_{i+1}(\mu, \nu)I_i(\lambda, \mu)E_{i+1}(\lambda, \mu) \end{aligned} \tag{2.29}$$

$$b_i(\lambda,\mu)I_{i+1}(\lambda,\nu)b_i(\mu,\nu)$$
$$=\{m(b_i(\lambda,\mu)+l^{-1}E_i(\lambda,\mu)\}+I_i(\lambda,\mu)I_{i+1}(\lambda,\nu)I_i(\mu,\nu) \tag{2.30}$$
$$b_{i+1}(\mu,\nu)I_i(\lambda,\mu)b_{i+1}(\lambda,\mu)$$
$$=\{m(b_{i+1}(\mu,\nu)+l^{-1}E_{i+1}(\mu,\nu)\}+I_{i+1}(\mu,\nu)I_i(\lambda,\nu)I_{i+1}(\lambda,\mu). \tag{2.31}$$

Equations (2.13)–(2.31) define the coloured B-WA (CB-WA) which will be useful in the Yang–Baxterization of CBGR for G [14, 15]. Note that when $\lambda=\mu=\nu$, the CB-WA reduces to the B-WA.

3. Link polynomials related to CBWA

As pointed out in [16, 17], there exists a new type of link polynomial other than the Jones–Kauffman link polynomial [18]. This new kind of link polynomial was coined by Lee and Kauffman the Alexander–Conway link polynomial [16]. We have found that some non-standard BGRs associated with C_n and D_n (which are not diagonalizable) lead to the Alexander–Conway link polynomials (ACLP) [14, 19]. In [1], Murakami presented the sufficient conditons under which ACLP can be constructed, namely, if the enhanced Yang–Baxter operators Y related to BGR satisfy the redundancy conditions then ACLP can be obtained in a similar way to that in deriving the Jones–Kauffman link polynomials. Furthermore, it was pointed out that any Y related to B-WA does definitely satisfy the redundancy conditions [19]. This is because for any B-WA the diagrammatic relation of figure 1 holds. In figure 1, $\bar{B}_n$ stands for a block formed by

$$\{I, b_1^{\pm}, b_2^{\pm}, \ldots, b_n^{\pm 1}\}$$

that will be closed to form a knot or link and

$$\bar{A}_n=\{\bar{b}_n, \bar{E}_n|\bar{E}_n=m^{-1}(\bar{b}_n+\bar{b}_n^{-1})-I\}$$

with b_i ($\bar{b}_n$ without colour) being the braiding between the ith and $(i+1)$th strings.

Figure 1 can be written in the form

$$\bar{B}_n=\bar{B}_{n-1}+\bar{B}_{n-1}\bar{A}_n\bar{B}_{n-1}. \tag{3.1}$$

In [19] we have shown that any B-WA for which (3.1) holds leads to the satisfaction of the redundancy conditions. Now we shall show that the same statement holds also for CB-WA.

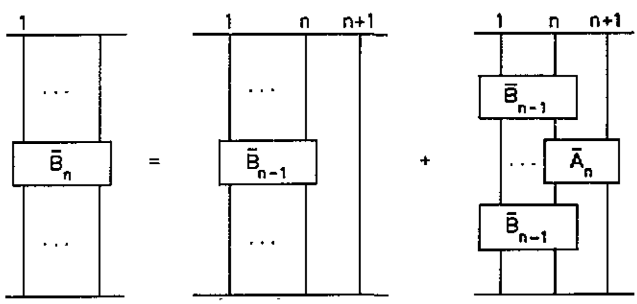


Figure 1. Illustration of equation (3.1).

Statement. If b_n obey CB-WA then we have

$$B_n = B_{n-1} + B_{n-1}A_nB_{n-1} \tag{3.2}$$

where B_n stands for a coloured braiding block formed by $\{I(\lambda,\mu), b_1^{\pm 1}(\lambda,\mu), \ldots, b_n^{\pm 1}(\lambda,\mu)\}$ and

$$A_n = \{E_n(\lambda,\mu)I_n(\mu,\lambda), b_n(\lambda,\mu)I_n(\mu,\lambda)\}. \tag{3.3}$$

It is emphasized that the braid block B_n in (3.2) should be closed to form a coloured knot or link. Diagrammatically (3.2) can be illustrated as in figure 2, where the role of $I_n(\mu,\lambda)$ is to interchange the colours λ and μ so that the graph can form a coloured knot or link. To prove (3.2) we first prove the following lemma.

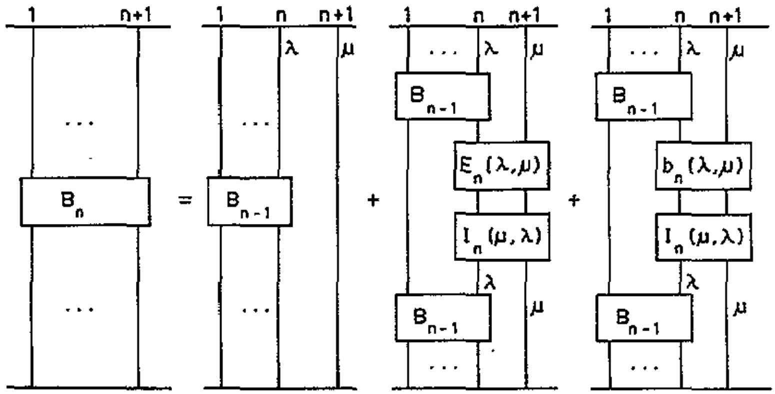


Figure 2. Illustration of equation (3.2).

Lemma. If a coloured BGR obeys CB-WA then for any open braiding block B_n' (not necessarily closed to form a knot or link) the following relation holds:

$$B_n' = B_{n-1}' + B_{n-1}'C_nB_{n-1}' \tag{3.4}$$

where C_n represents elements of the set

$$\{b_n(\lambda,\mu), E_n(\lambda,\mu), I_n(\lambda,\mu), E_n(\lambda,\mu)I_n(\mu,\lambda), b_n(\lambda,\mu)I_n(\mu,\lambda)\}. \tag{3.5}$$

Proof. First we note that any braiding block can be formed by

$$\sum_{\{\alpha\}}\prod_{\{\iota\}} b_i^a E_j^b I_k^c (E_lI_l)^d (b_mI_m)^e \in B_n' \tag{3.6}$$

for CB-WA where

$$\{\alpha\} \equiv \{a, b, c, d, e\} = \{0, 1\}$$

and

$$\{\iota\} \equiv \{i, j, k, l, m\} = \{1, 2, \ldots, n-1\}.$$

Equation (3.6) can easily be verified by using definitions of $E_i(\lambda,\mu)$ and $I_j(\mu,\lambda)$ and the CB-WA structure; the latter guarantees the set $\{\alpha\}$ runs over $\{0, 1\}$ only.

Next we show that any element C_i of the coloured set formed from (3.5) satisfies the relation

$$C_iC_{i+1}C_i = C_{i+1}C_iC_{i+1}. \tag{3.7}$$

Equation (3.7) can be verified by exhausting all of the possibilities. For instance, in LHS of (3.7) one of the possibilities can be formally expressed by

$$\begin{pmatrix} b_i(\mu,\nu) \\ E_i(\mu,\nu) \\ I_i(\mu,\nu) \end{pmatrix} \otimes \begin{pmatrix} b_{i+1}(\lambda,\nu) \\ E_{i+1}(\lambda,\nu) \\ I_{i+1}(\lambda,\nu) \end{pmatrix} \otimes \begin{pmatrix} b_i(\lambda,\mu) \\ E_i(\lambda,\mu) \\ I_i(\lambda,\mu) \end{pmatrix}. \tag{3.8}$$

We first fix, for example, $I_i(\mu,\nu)$ and let any element in the second space and in the third space be $\Delta'_{i+1}(\lambda,\nu)$ and $\Delta''_i(\lambda,\mu)$, respectively, then using (2.14), (2.17) and (2.21) we get

$$\begin{aligned} I_i(\mu,\nu)&\Delta'_{i+1}(\lambda,\nu)\Delta''_i(\lambda,\mu) \\ &= I_{i+1}(\lambda,\mu)I_{i+1}(\mu,\lambda)I_i(\mu,\nu)\Delta'_{i+1}(\lambda,\nu)\Delta''_i(\lambda,\mu) \\ &= I_{i+1}(\lambda,\mu)\Delta'_{i+1}(\mu,\lambda)I_i(\mu,\nu)I_{i+1}(\lambda,\nu)\Delta''_i(\lambda,\mu) \\ &= I_{i+1}(\lambda,\mu)\Delta'_{i+1}(\mu,\lambda)I_{i+1}(\lambda,\mu)\Delta''_i(\lambda,\nu)I_{i+1}(\mu,\nu). \end{aligned} \tag{3.9}$$

Since $\Delta'_{i+1}(\lambda,\nu)$ and $\Delta''_i(\lambda,\mu)$ can be b, E or I, equation (3.9) becomes

$$\begin{aligned} I_{i+1}(\lambda,\mu)&\Delta'_{i+1}(\mu,\lambda)I_{i+1}(\lambda,\mu)\Delta''_i(\lambda,\nu)I_{i+1}(\mu,\nu) \\ &= \Delta'_{i+1}(\lambda,\mu)\Delta''_i(\lambda,\nu)I_{i+1}(\mu,\nu) \in C_{i+1}(\lambda,\mu)C_i(\lambda,\nu)C_{i+1}(\mu,\nu) \end{aligned} \tag{3.10}$$

where (2.14), (2.16) and (2.23) have been used. Similarly

$$\begin{aligned} \Delta'_i(\mu,\nu)&I_{i+1}(\lambda,\nu)\Delta''_i(\lambda,\mu) \\ &= I_i(\mu,\nu)I_{i+1}(\lambda,\nu)\Delta'_i(\lambda,\mu)\Delta''_i(\mu,\lambda)I_i(\lambda,\mu) \\ &= I_i(\mu,\nu)I_{i+1}(\lambda,\nu)\{\Delta'_i(\lambda,\mu)\Delta''_i(\mu,\lambda)I_i(\lambda,\mu)\}. \end{aligned}$$

Noting that the term inside the parentheses is till an element $\Delta'''_i(\lambda,\nu) \in C_i(\lambda,\mu)$ we thus have

$$\Delta'_i(\mu,\nu)I_{i+1}(\lambda,\nu)\Delta''_i(\lambda,\mu) = I_{i+1}(\lambda,\mu)\Delta'''_i(\lambda,\nu)I_{i+1}(\mu,\nu)$$

which belongs to $C_iC_{i+1}C_i = C_{i+1}C_iC_{i+1}$. In a similar way, the rest of (3.7) in the form of (3.8) can be completely verified by using the CB-WA.

Another possibility of equation (3.7) is

$$\underset{\Delta'(\mu,\nu)}{\begin{pmatrix} b_i(\mu,\nu) \\ E_i(\mu,\nu) \\ I_i(\mu,\nu) \end{pmatrix}} \otimes \underset{\Delta''_{i+1}(\lambda,\nu)I_{i+1}(\nu,\lambda)}{\begin{pmatrix} E_{i+1}(\lambda,\nu)I_{i+1}(\nu,\lambda) \\ b_{i+1}(\lambda,\nu)I_{i+1}(\nu,\lambda) \end{pmatrix}} \otimes \underset{\Delta'''_i(\lambda,\mu)}{\begin{pmatrix} b_i(\lambda,\mu) \\ E_i(\lambda,\mu) \\ I_i(\lambda,\mu) \end{pmatrix}} \tag{3.11}$$

For instance making use of CB-WA relations and after some calculation, we have

$$\Delta'_i(\mu,\nu)\{\Delta''_{i+1}(\lambda,\nu)I_{i+1}(\nu,\lambda)\}\Delta'''_i(\lambda,\mu) = \Delta'_{i+1}(\lambda,\mu)\{\Delta''_i(\lambda,\nu)I_i(\nu,\lambda)\}\Delta''_{i+1}(\mu,\nu) \tag{3.12}$$

which coincides with (3.7). In a similar manner, we exhaust all the possibilities in the LHS and RHS of (3.7) to show its validity by using the relations of CB-WA.

With the help of (3.7), we can literally repeat all the arguments made in [19] for the non-coloured B-WA to prove the coloured extension (3.4). To save space we omit the details and give only a graphic illustration.

A part of the proof can be illustrated diagrammatically by assuming (3.4) is true for n and then showing its validity for $n+1$ through induction. For $n=2$, equation (3.4) is simply (3.7) itself. For any n, we assume that (3.4) is valid. Because, for any braiding block B'_n, there should be

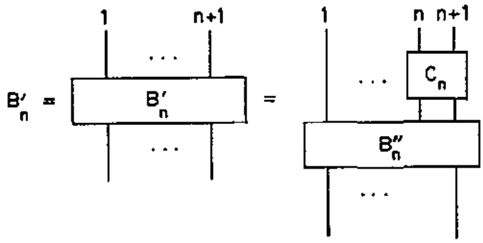


Figure 3.

where both B'_n and B''_n belong to same braiding block formed by $(I, b_1, b_2^{\pm 1}, \ldots, b_n^{\pm 1})$. Following the Markov properties [13], this becomes

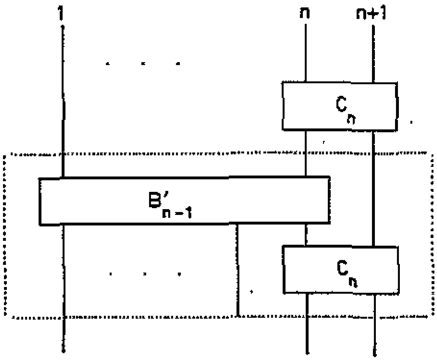


Figure 4

where the dotted block represents the B''_n in figure 3.

Assuming (3.4) is valid for $n-1$ the non-trivial part of figure 4 has the form

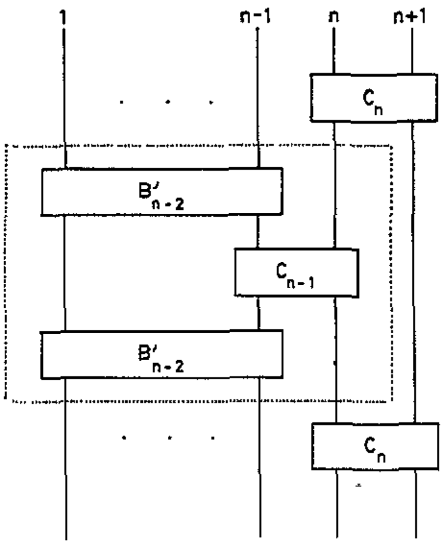
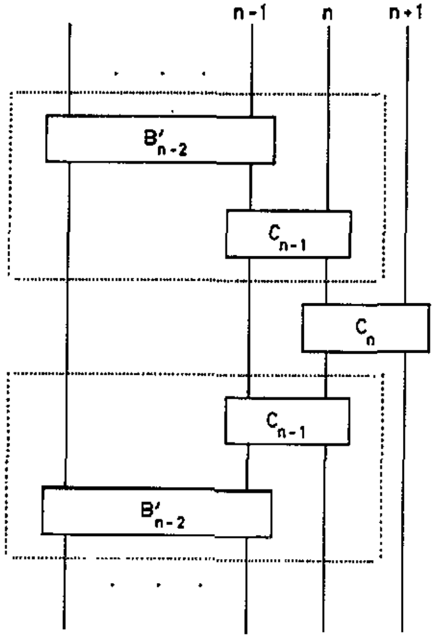


Figure 5

then because of (3.7) we obtain



$$= B'_{n-1} C_n B'_{n-1} \tag{3.13}$$

Figure 6

where the fact that the dotted blocks in the LHS belong to B'_{n-1} has been used.

If the considered braiding blocks are closed to form a knot or link then (3.4) gives (3.2) with (3.3). This is because only strings with the same colours can be closed. It is natural to have equation (3.4) since CB-WA preserves the basic algebraic structure as B-WA. The only difference is that all strings are coloured and entangled by $I_i(\lambda, \mu)$, $b_i(\lambda, \mu)$ and $E_i(\lambda, \mu)$. When the algebra is extended to take colours into account, there is no essential change in the topology.

On the basis of (3.4), we can follow Murakami [1] to find Alexander–Conway link polynomials, namely, if there exists H such that

$$\mathrm{tr}_{n+1}\{E_n(\lambda, \mu) I_n(\mu, \lambda)(\underbrace{I \otimes \ldots \otimes I}_{n} \otimes H)\} = kI^{\otimes n} \tag{3.14}$$

$$\mathrm{tr}_{n+1}\{b_n(\lambda, \mu) I_n(\mu, \lambda)(\underbrace{I \otimes \ldots \otimes I}_{n} \otimes H)\} = k' I^{\otimes n} \tag{3.15}$$

where k and k' are constants and the trace tr_{n+1} is taken for the $(n+1)$th space only; then when

$\mathrm{tr}\, H = 0$	this leads to redundancy, i.e. to ACLP
$\mathrm{tr}\, H \neq 0$	this leads to the usual coloured Jones–Kauffman link polynomials.

4. Non-standard $\check{R}(\lambda, \mu)$ for D_2

The general coloured extension of non-standard $\check{R}(\lambda, \mu)$ for G is tremendously complicated because the calculation is strongly model-dependent.

In order to make comparison between equations (1.2) and (1.3), we take D_2 as an example. As pointed out in [14, 20], a non-trivial q-solution of BGR for D_2 can be obtained by solving braid relations. The classical Lie algebra $D_2 \simeq \mathrm{SU}(2) \otimes \mathrm{SU}(2)$ is decomposable. After 'q-deformation' the representations for SU(2) can be taken to be either standard or non-standard, namely, there exist four trivial possibilities. However, it turns out that there is a fifth q-solution which is neither decomposable nor diagonalizable. The non-trivial non-standard BGR for D_2 possesses the form [14, 20]

$$\check{R}_{D_2} = \text{block diag}(A_1, A_2, A_3, A_4, A_3, A_2, A_1)$$

$$A_1 = q \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & w \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -q^{-1} & 0 \\ 1 & 0 & w \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & -q^{-1} \\ 0 & 0 & -q & -iw \\ 0 & -q & 0 & -iw \\ q^{-1} & -iw & -iw & 2w \end{bmatrix} \qquad (w = q - q^{-1}). \tag{4.1}$$

Here, we emphasize that for SU(2) case the non-standard solutions are related to the representations of quantum algebra $\mathrm{SU}_q(2)$ for q at root of unity [3, 10]. What we are concerned with is whether the non-standard solution for D_2 possesses also the same property; the answer, as we shall see, is no. The condition leading to q being a root of unity comes from another source different from the SU(2) case. To see this point let us first solve (1.1) for D_2.

The calculation follows the same strategy as in deriving (4.1) but with the extension that all of the matrix forms for the solution are perserved and the unknown parameters are dependent on colours λ and μ. By labelling the $\check{R}(\lambda, \mu)$ matrix for D_2 in terms of

$$a, b, c, d \in [-\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}] \tag{4.2}$$

the general form of $\check{R}(\lambda, \mu)$ for D_2 is as given by (2.1). Without confusion in this section, we still use the notations of section (2) for D_2.

Substituting (4.1) into (1.1), and after tedious calculations (see appendix), we derive the following form of solutions for D_2

$$\check{R}(\lambda, \mu) = \text{block diag}\{A_1^c, A_2^c, A_3^c, A_4^c, A_3^{c'}, A_2^{c'}, A_1^{c'}\} \tag{4.3}$$

$$A_1^c = q \qquad A_1^{c'} = qX_+(\lambda)X_-(\lambda)Y_+(\mu)Y_-(\mu).$$

$$A_2^c = \begin{bmatrix} 0 & X_+(\lambda) \\ Y_+(\mu) & A_+(\lambda)B_+(\mu) \end{bmatrix} \qquad A_2^{c'} = \begin{bmatrix} 0 & X_+(\lambda)X_-(\lambda)Y_-(\mu) \\ X_-(\lambda)Y_+(\mu)Y_-(\mu) & A'_-(\lambda)B'_-(\mu) \end{bmatrix}$$

$$A_3^c = \begin{bmatrix} 0 & 0 & X_-(\mu) \\ 0 & -q^{-1}X_+(\lambda)Y_+(\mu) & 0 \\ Y_-(\mu) & 0 & A_-(\lambda)B_-(\mu) \end{bmatrix}$$

$$A_3^{c'} = \begin{bmatrix} 0 & 0 & X_+(\lambda)X_-(\lambda)Y_+(\mu) \\ 0 & -qX_-(\lambda)Y_-(\mu) & 0 \\ X_-(\lambda)Y_+(\mu)Y_-(\mu) & 0 & A'_+(\lambda)B'_+(\mu) \end{bmatrix} \tag{4.4}$$

$$A_4^c = \begin{bmatrix} 0 & 0 & 0 & q^{-1}X_+(\lambda)X_-(\mu) \\ 0 & 0 & -qX_-(\lambda)Y_+(\mu) & \mathrm{i}X_-(\lambda)Y_-^{-1}(\mu)A_+(\lambda)B'_-(\mu) \\ 0 & -qX_+(\lambda)Y_-(\mu) & 0 & \mathrm{i}X_+(\lambda)Y_+^{-1}(\mu)A_-(\lambda)B'_+(\beta) \\ q^{-1}Y_+(\mu)Y_-(\mu) & L & M & W^{(-3/2,3/2)}(\lambda, \mu) \end{bmatrix}$$

where

$$L \equiv \mathrm{i} Y_+(\mu) X_-^{-1}(\lambda) A'_-(\lambda) B_+(\mu) \qquad M \equiv \mathrm{i} X_+^{-1}(\lambda) Y_+(\mu) A'_+(\lambda) B_-(\mu)$$

$$W^{(-3/2,3/2)}(\lambda,\mu) = X_-(\mu)(A'_-(\mu))^{-1} B_-(\mu)\{X_+^{-1}(\lambda) X_+(\mu) Y_+(\mu) A'_+(\lambda) A_+(\lambda) + X_-^{-1}(\lambda) A'_-(\lambda) A_-(\lambda)\} \tag{4.5}$$

and $+\equiv\frac{1}{2}$, $-\equiv-\frac{1}{2}$ for the subscripts of matrix elements in (4.4)–(4.5).

To satisfy (1.1) the parameters appearing in (4.4) and (4.5) admit the following two types of solutions

(1) q can be an arbitrary complex number (generic) and

$$Y_a = X_a^{-1} \qquad A_a(\lambda) B_a(\lambda) = A'_a(\lambda) B'_a(\lambda) = w = (q - q^{-1}) \tag{4.6}$$

where $a = \pm$ and repeated indices are not summed. In (4.6), q, $Y_\pm$, $A_\pm$ and $A'_\pm$ are free parameters.

(2) $q^4 = 1$ and for $a = \pm$

$$\begin{aligned} &X_+(\lambda) Y_+(\lambda) = X_-(\lambda) Y_-(\lambda) \neq 1 \\ &A'_a(\lambda) B'_a(\lambda) = X_+(\lambda) Y_+(\lambda)\{q X_+(\lambda) Y_+(\lambda) - q^{-1}\} \\ &A_a(\lambda) B_a(\lambda) = \{q - q^{-1} X_+(\lambda) Y_+(\lambda)\}. \end{aligned} \tag{4.7}$$

Obviously solution (4.6) is the straightforward colour extension of the non-standard solution (4.1) since when $\lambda = \mu$ (by taking $X = Y = 1$ and $A'_\pm = -A_\mp$, $B'_\pm = -B_\mp$), it reduces to the usual non-standard solution (4.1).

The solution (4.7) is much more interesting because it is not simply the coloured extension of (4.1), but is due to the consequence of the colour-dependent constraints. To satisfy (1.1), the stringent colour-dependent constraints require the parameter q to be a root of unity. This phenomenon has never been encountered before. Let us recall the $\mathrm{SU}_q(2)$ case. As pointed out in [3, 10], the non-standard coloured solutions of $\check{R}(\lambda, \mu)$ associated with SU(2) are related to representations of quantum algebra $\mathrm{SU}_q(2)$ for q a root of unity (in previous work [10, 16] it is expressed by ω, $\omega^N = 1$), whereas the colours λ and μ are additional parameters allowed by the quantum double [3, 10]. In other words, for $\mathrm{Su}_q(2)$ the colour 'degree' is a separated one, for example, the minus sign appearing before $q^{-1} X(\lambda) Y(\mu)$ in (1.3) just stands for the $\omega(=-1)$ in the theory [3, 5, 10]. Therefore in the non-standard solutions for SU(2) the behaviour of q at the root of unity is independent of any colour-constraints. However the picture of q at the root of unity associated with D_2 comes purely from the colour-constraints. This phenomena is quite new.

Since the general R-operator theory for G given by Rosso *et al* [21] is in the abstract form we could not make comparison between the solution and quantum algebra. We hope that it can be interpreted in terms of the present quantum double theory and look for more general solutions for $\check{R}(\lambda, \mu)$ with $q^p = 1$ being constrained by colours.

Acknowledgments

M L Ge is indebted to the Department of Physics of National University of Singapore for their hospitality. This work is in part supported by NFS of China and research grant RP910684 from the National University of Singapore.

Appendix

Derivation of solutions to equations (4.3)–(4.7).

By substituting the general form of $\check{R}(\lambda, \mu)$ for D_2 as given by (2.1) into (1.1), we first obtain

$$p^{(3/2,\pm)}(\lambda, \mu)u_{3/2}(\lambda, \nu) = p^{(3/2,\pm)}(\lambda, \nu)u_{3/2}(\lambda, \mu) \tag{A.1}$$

and if we take

$$u_{3/2}(\lambda, \mu) = q \tag{A.2}$$

then

$$p^{(3/2,\pm)}(\lambda, \mu) = X_\pm(\lambda). \tag{A.3}$$

Similarly by solving the relations involving $u_{3/2}(\lambda, \mu)$, $u_\pm(\lambda, \mu)$, $p^{(\pm 3/2,\pm)}(\lambda, \mu)$ and $p^{(\pm,\pm 3/2)}(\lambda, \mu)$, we obtain (omitting the arguments λ and μ)

$$\begin{aligned}
&u_{3/2} = q \qquad u_\pm = -q^{-1}X_\pm(\lambda)Y_\pm(\mu)\\
&u_{-3/2} = qX_+(\lambda)X_-(\lambda)Y_+(\mu)Y_-(\mu)\\
&p^{(\pm,3/2)} = Y_\pm(\mu) \qquad p^{(3/2,\pm)} = X_\pm(\lambda)\\
&p^{(-3/2,\pm)} = qX_\pm(\lambda)Y_{-3/2}(\mu) \qquad p^{(\pm,-3/2)} = qX_{-3/2}(\lambda)Y_\pm(\mu)\\
&p^{(\pm,\mp)} = -qX_\mp(\lambda)Y_\pm(\mu).
\end{aligned} \tag{A.4}$$

Because $W^{(-,+)} = 0$ and from

$$W^{(\pm,3/2)}(\lambda, \mu)W^{(\pm,3/2)}(\mu, \nu) = \{q - q^{-1}X_\pm(\mu)Y_\pm(\mu)\}W^{(\pm,3/2)}(\lambda, \nu)$$

it follows

$$X_+(\mu)Y_+(\mu) = X_-(\mu)Y_-(\mu). \tag{A.5}$$

Combining with other relations, we then have

$$\begin{aligned}
&W^{(\pm,3/2)}(\lambda, \mu) = A_\pm(\lambda)B_\pm(\mu)\\
&W^{(-3/2,\pm)}(\lambda, \mu) = A'_\pm(\lambda)B'_\pm(\mu)
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
&A_\pm(\mu)B_\pm(\mu) = (q - q^{-1}X_+(\mu)Y_+(\mu))\\
&A'_\pm(\mu)B'_\pm(\mu) = (qX_+(\mu)Y_+(\mu) - q^{-1})X_+(\mu)Y_+(\mu).
\end{aligned} \tag{A.7}$$

By solving the relations including $p^{(\pm 3/2,\pm)}$, $q^{(\pm 3/2,\pm)}$ and $w^{(\pm,\pm 3/2)}$, we obtain

$$q^{(-3/2,+)}(\lambda, \mu) = \mathrm{i}\,Y_-(\mu)X_-^{-1}(\lambda)A'_-(\lambda)B_+(\mu) \tag{A.8}$$

$$q^{(+,-3/2)}(\lambda, \mu) = \mathrm{i}X_-(\lambda)Y_-^{-1}(\mu)A_+(\lambda)B'_-(\mu) \tag{A.9}$$

and

$$B'_-(\lambda)A'_+(\lambda) = X_+(\lambda)Y_-(\lambda)A_-(\lambda)B_+(\lambda) \tag{A.10}$$

$$B'_+(\mu)A'_-(\mu) = X_-(\mu)Y_+(\mu)A_+(\mu)B_-(\mu). \tag{A.11}$$

Putting the above relations together, we have

$$\{qX_+(\lambda)Y_+(\lambda) - q^{-1}\}^2 = \{q - q^{-1}X_+(\lambda)Y_+(\lambda)\}^2 \tag{A.12}$$

from which it follows that there exist two solutions:

$$(1) \qquad q^4=1 \qquad X_+(\lambda)Y_+(\lambda)\neq 1 \tag{A.13}$$

$$(2) \qquad q \text{ is generic, but } X_+(\lambda)Y_+(\lambda)=\varepsilon \qquad \varepsilon^2=1. \tag{A.14}$$

From

$$B_1'(\lambda)A_+'(\lambda)=X_+(\lambda)Y_-(\lambda)A_-(\lambda)B_+(\lambda)$$

$$B_-'(\lambda)q^{(-3/2,-)}(\lambda,\mu)=\mathrm{i}\,Y_+(\lambda)Y_+(\mu)A_-(\lambda)B_-(\mu)B_+(\lambda)$$

it follows

$$q^{(-3/2,-)}(\lambda,\mu)=\mathrm{i}X_{+}^{-1}(\lambda)Y_+(\mu)A_+'(\lambda)B_-(\mu). \tag{A.15}$$

By virtue of the relation

$$W^{(-3/2,-)}(\lambda,\mu)p^{(-,+)}(\lambda,\nu)W^{(-3/2,+)}(\mu,\nu)=q^{(-,-3/2)}(\mu,\nu)u_{-3/2}(\lambda,\nu)q^{(-3/2,+)}(\lambda,\mu)$$

we get

$$q^{(-,-3/2)}(\lambda,\mu)=\mathrm{i}X_+(\lambda)Y_{+}^{-1}(\mu)A_-(\lambda)B_+'(\mu). \tag{A.16}$$

Employing the relation

$$W^{(-3/2,3/2)}(\lambda,\mu)u_{3/2}(\lambda,\nu)q^{(-3/2,+)}(\mu,\nu)+q^{(-3/2,-)}(\lambda,\mu)W^{(+,3/2)}(\lambda,\mu)p^{(-,+)}(\mu,\nu)$$
$$=u_{3/2}(\mu,\nu)q^{(-3/2,+)}(\lambda,\nu)W^{(-,3/2)}(\lambda,\mu) \tag{A.17}$$

and other corresponding relations, we finally arrive at

$$W^{(3/2,3/2)}(\lambda,\mu)$$
$$=X_-(\mu)A_-'^{-1}(\mu)B_-(\mu)\{X_{+}^{-1}(\lambda)X_+(\mu)Y_+(\mu)A_+'(\lambda)A_+(\lambda)$$
$$+X_{-}^{-1}(\lambda)A_-'(\lambda)A_-(\lambda)\}.$$

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